J384. In triangle ABC, $A < B < C$. Prove that

$$\cos\frac{A}{2}\csc\frac{B-C}{2}+\cos\frac{B}{2}\csc\frac{C-A}{2}+\cos\frac{C}{2}\csc\frac{A-B}{2}<0.$$

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Solution by Arkady Alt, San Jose, CA, USA

Note that

$$\cos\frac{A}{2}\csc\frac{B-C}{2}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{A}{2}\sin\frac{B-C}{2}}=\frac{\sin A}{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}=\frac{\sin A}{\sin B-\sin C}=\frac{a}{b-c}.$$

Thus, the inequality is equivalent to $\sum_{\text{cyc}}\frac{a}{b-c}<0$. Since $A<B<C$, we know $a<b<c$, and so

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=\frac{b}{c-a}-\frac{c}{b-a}-\frac{a}{c-b}=\left(\frac{b}{c-a}-\frac{a}{c-b}\right)-\frac{c}{b-a}=-\frac{(b-a)(a+b-c)}{(c-b)(c-a)}-\frac{c}{b-a}<0.$$

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Pamplona, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Utsab Sarkar, West Bengal, India; Anderson Torres, Sao Paulo, Brazil; Polyahedra, Polk State College, USA.